Function approximation with one-bit Bernstein and neural networks

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References

Figure: Sinan Güntürk

References:

- Approximation with one-bit polynomials in Bernstein form.
  arXiv:2112.09183
- Approximation of functions with one-bit neural networks.
  arXiv:2112.09181

Can also be found on my webpage:

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Overview

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Main question

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- A quantized NN is one where all non-zero parameters (weights and biases) must be selected from a set of discrete values ("alphabet").

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  For example, a "high-resolution" arithmetic progression

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- A coarsely quantized NN uses an alphabet with a small collection of values.

  For example, a one-bit alphabet,

$$\mathcal{A}_1 := \{\pm 1\}.$$

Motivation 1: Practical issue

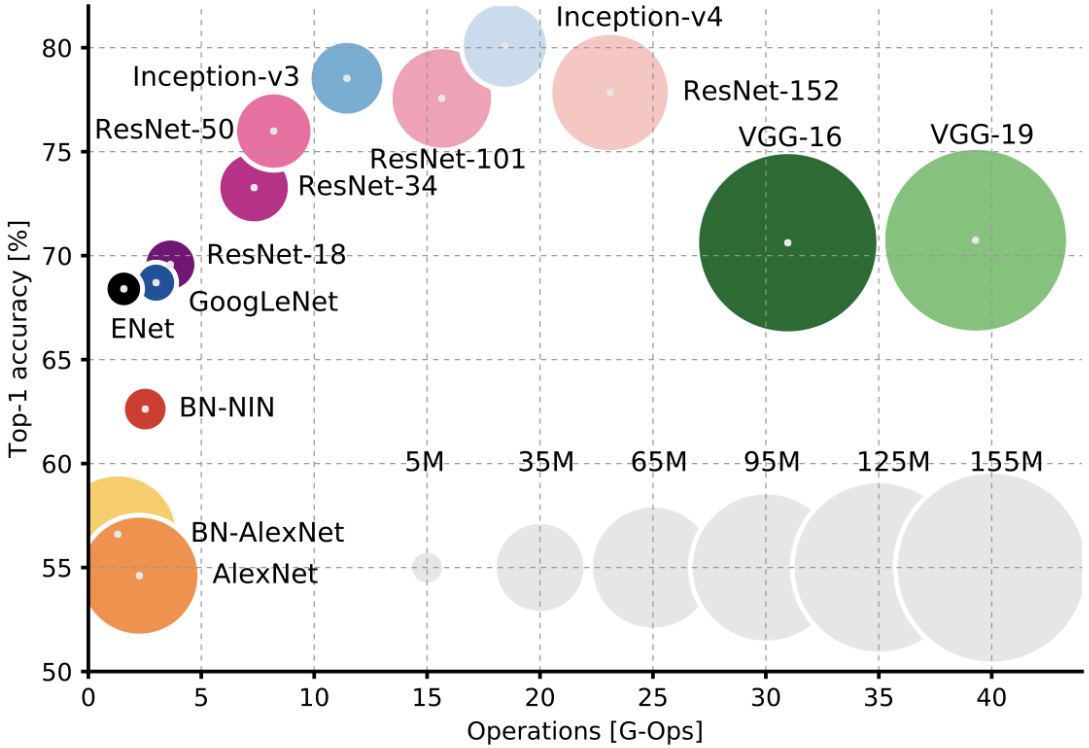


Figure: Memory footprint of some networks [Canziani, Paszke, Culurciello 16]

How to transfer huge networks onto low power devices (phones, cars, tablets, etc.)?

Survey article [Yunhui Guo 18]

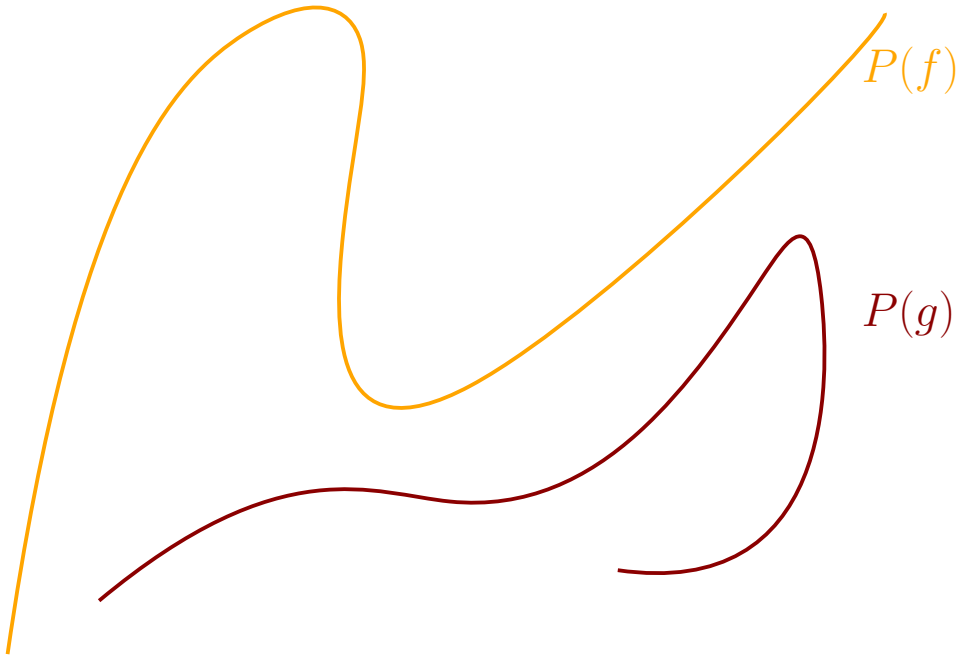
$$\begin{aligned} \text{Quantize the network} &\longrightarrow \text{low-bit representation} \\ &\longrightarrow \text{cheaper storage cost and faster forward pass} \\ &\longrightarrow \text{less energy consumption} \end{aligned}$$

Use of sophisticated quantization techniques: [Lybrand, Saab 21], [Ashbrock, Powell 21]

Motivation 2: Over-parameterization

For each f, look at the parameters that represent f,

$$P(f) = \{\theta \in \mathbb{R}^D : NN_\theta = f\}$$



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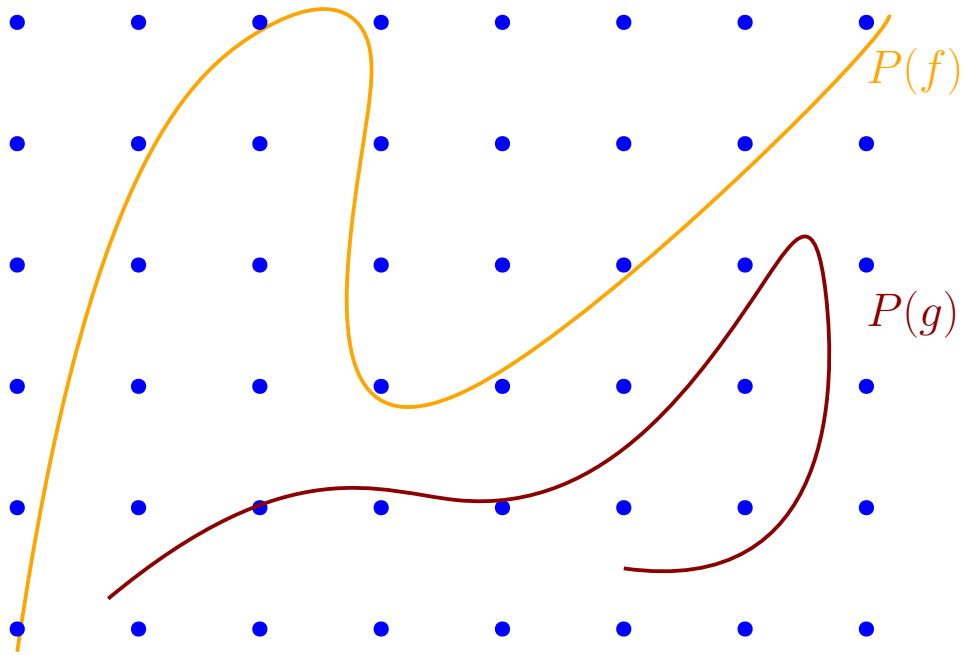

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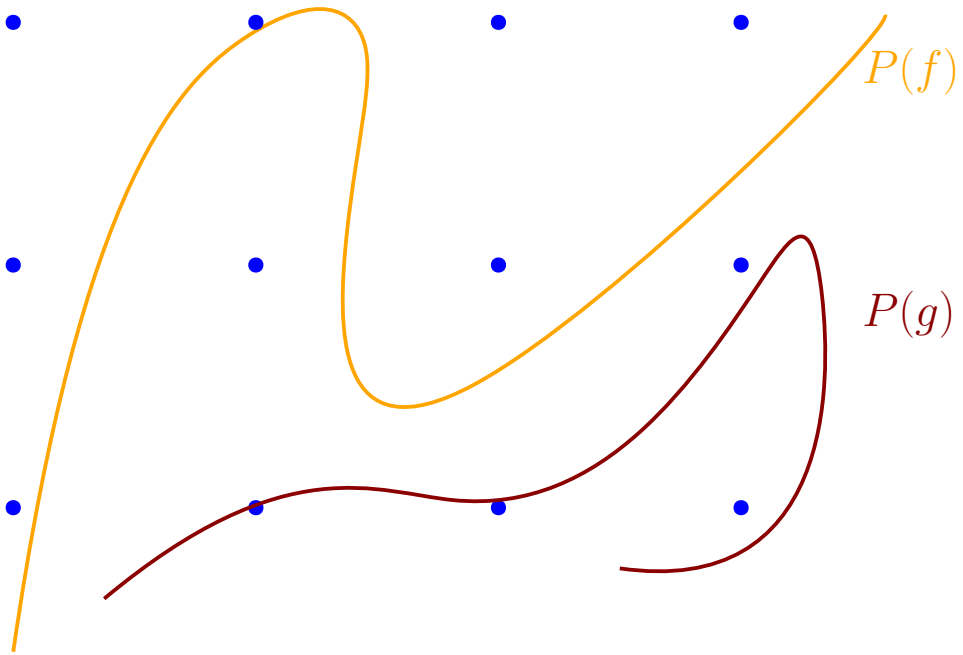

A more technical viewpoint from approximation theory

What makes the main question difficult to answer?

Numerous results for approximation by neural networks (incomplete list of course):

[Cybenko 89], [Barron 94], [Yarotsky 17], [Shaham, Cloninger, Coifman 18], [Bolcskei, Grohs, Kutyniok, Petersen 19], [Daubechies, DeVore, Foucart, Hanin 21], [Lu, Shen, Yang, Zhang 21]

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Main strategy of the above approximation strategies:
For fixed f,

$$f \approx \sum_k a_k \phi_k$$

- $\{\phi_k\}_k$ are implementable by a neural network with carefully chosen parameters, e.g., local polynomials, ridge functions, wavelets, Fourier
- $\{a_k\}_k \subseteq \mathbb{R}$ depend on f, e.g., basis or frame coefficients

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Approach needs to be modified when we consider coarsely quantized networks!

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Our wish list

Find a suitable set of functions $\{\phi_k\}_k$ such that:

- **Approximation.** Linear combinations of $\{\phi_k\}_k$ can efficiently approximate large function classes.
- **Implementation.** The $\{\phi_k\}_k$ can be implemented by a coarsely quantized neural network.

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$$\sum_k a_k \phi_k - \sum_k \sigma_k \phi_k \quad \text{is small.}$$

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Three term decomposition:

$$f - f_{NN} = \underbrace{\left(f - \sum_k a_k \phi_k\right)}_{\text{approximation by } \phi \text{ error}} + \underbrace{\left(\sum_k a_k \phi_k - \sum_k \sigma_k \phi_k\right)}_{\text{quantization error}} + \underbrace{\left(\sum_k \sigma_k \phi_k - f_{NN}\right)}_{\text{implementation error}}.$$

Function approximation with integer Bernstein

Kantorovich: If $f\colon [0,1] \to \mathbb{R}$ is continuous and $f(0)$ and $f(1)$ are integers, then the function

$$B_n^*(f)(x) := \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] x^k (1-x)^{n-k},$$

converges uniformly to f as $n \to \infty$. Here, $[t]$ is rounding t to the nearest integer.

Proof can be found in Chapter 2.4 of *Constructive Approximation: Advanced Problems* by Lorentz, v Golitschek, and Makovoz.

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- Implies that polynomials with integer coefficients are dense in the space of continuous functions.
- Main drawback is that if $\|f\|_\infty \leq 1$, then

$$\left| \left[f\left(\frac{k}{n}\right) \binom{n}{k} \right] \right| \quad \text{could be as big as} \quad 2^n,$$

so not helpful for coarse quantization.

One-bit Bernstein?

Bernstein polynomials of degree n: for each $0 \leq k \leq n$,

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}.$$

Berstein: For any $f \in C([0,1])$,

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Question: For any $f \in C([0,1])$ and $\epsilon > 0$, is there an integer n and $\{\sigma_k\}_{k=0}^n$ such that $\sigma_k \in \{\pm 1\}$ and

$$\left\| f - \sum_{k=0}^{n} \sigma_k p_{n,k} \right\|_\infty \leq \epsilon?$$

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Question: For any $f \in C([0,1])$ such that $\|f\|_\infty \leq 1$ and $\epsilon > 0$, is there an integer n and $\{\sigma_k\}_{k=0}^n$ such that $\sigma_k \in \{\pm 1\}$ and for all x away from the endpoints,

$$\left| f(x) - \sum_{k=0}^{n} \sigma_k p_{n,k}(x) \right| < \epsilon?$$

A first approximation result

Theorem

For any Lipschitz continuous f with $\|f\|_\infty \le 1$ and integer n, there exist $\{\sigma_k\}_{k=0}^n \subset \{\pm 1\}$, such that

$$\left| f(x) - \sum_{k=0}^{n} \sigma_k p_{n,k}(x) \right| \lesssim \frac{1 + |f|_{Lip}}{1 + \sqrt{nx(1-x)}}.$$

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- By density, can extend to continuous functions.
- In comparison to Kantorovich, our construction uses significantly fewer bits.
- Implies a L^p estimate: for all $p \in [1, 2)$,

$$\left\| f - \sum_{k=0}^{n} \sigma_k p_{n,k} \right\|_{L^p([0,1])} \lesssim \frac{1 + |f|_{Lip}}{\sqrt{n}}.$$

A constructive proof

The sequence of ± 1 are chosen through an algorithm called first order $\Sigma\Delta$.
Set $a_k = f(k/n)$ and we find σ such that it satisfies a finite difference equation

$$a_k - \sigma_k = (\Delta u)_k := u_k - u_{k-1}.$$

Can always be done with a $\|u\|_\infty \lesssim 1$ due to the assumption that $\|f\|_\infty \leq 1$. Explicitly,

$$u_k := u_{k-1} + a_k - \sigma_k, \quad \sigma_k := \text{sign}(a_k + u_{k-1}).$$

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Bound the total error:

$$\begin{aligned}
&\left| f(x) - \sum_{k=0}^{n} \sigma_k p_{n,k}(x) \right| \\
&\leq \left| f(x) - \sum_{k=0}^{n} a_k p_{n,k}(x) \right| + \left| \sum_{k=0}^{n} (a_k - \sigma_k) p_{n,k}(x) \right| \\
&\lesssim \frac{|f|_{Lip}}{\sqrt{n}} + \left| \sum_{k=0}^{n} (\Delta u)_k \, p_{n,k}(x) \right|
\end{aligned}$$

Defining $p_{n,k+2} := 0$,

$$\begin{aligned}
\left| \sum_{k=0}^{n} (\Delta u)_k \, p_{n,k}(x) \right| &= \left| \sum_{k=0}^{n} u_k \left(p_{n,k}(x) - p_{n,k+1}(x) \right) \right| \\
&\leq \|u\|_\infty \sum_{k=0}^{n} |p_{n,k}(x) - p_{n,k+1}(x)| \quad \text{(Frame variation)} \\
&= \frac{\|u\|_\infty}{(n+1)x(1-x)} \sum_{k=0}^{n} |(k+1) - (n+1)x| \, p_{n+1,k+1}(x) \\
&= \frac{\|u\|_\infty}{(n+1)x(1-x)} \left(\sum_{k=0}^{n} |(k+1) - (n+1)x|^2 \, p_{n+1,k+1}(x) \right)^{1/2} \\
&\lesssim \frac{1}{\sqrt{nx(1-x)}}.
\end{aligned}$$

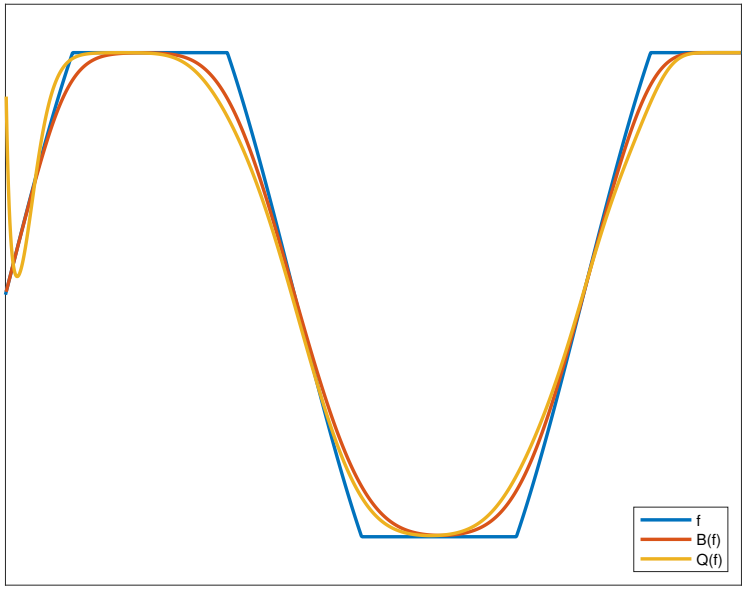
Frame variation 2 papers, [Benedetto, Powell, Yilmaz, 06]

Simple demonstration

Consider the continuous function,

$$f(x) = \max\left(\min\left(\frac{3}{2}\sin(8x), 1\right), -1\right).$$

Plot of $B_n(f)$ and constructed quantized approximation, for $n = 100$.



Coefficients of quantized approximation (where $-1 \leftrightarrow 0$):

$$10101111111111111111111111111111011110101001000000$$

$$00000000000000000000100010011011011111111111111$$

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Technical difficulties when extending to higher dimensions

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high dimensionality $\longrightarrow$ spaces of smooth functions.

Multivariate Bernstein of order n: For each $k \in \mathbb{N}^d$ with $0 \leq k_j \leq n$,

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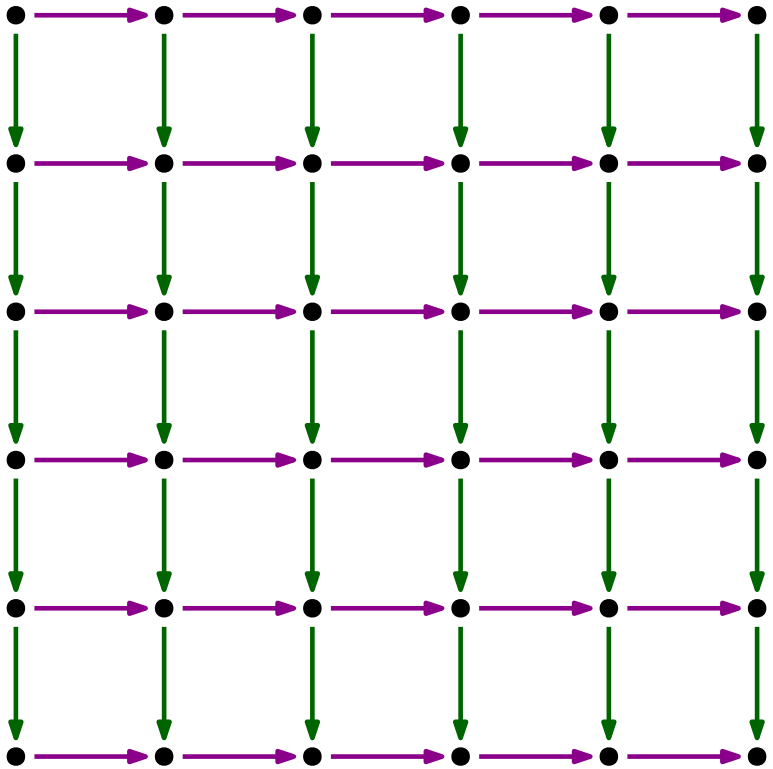
- High dimensionality is a major issue for $\Sigma\Delta$. No available $O(1)$-alphabet result for stable d dimensional $\Sigma\Delta$.
- Higher order smoothness is an issue do to this saturation result:
  If $f \in C^2([0,1])$, then

$$\lim_{n\to\infty} n(B_n(f)(x) - f(x)) = \frac{f''(x)x(1-x)}{2}.$$

  Bernstein polynomial of f is unable to exploit additional smoothness.

$\Sigma\Delta$ in higher dimensions

Say $d = 2$. How to order $\{p_{n,k}\}_{k_1=0,k_2=0}^{n}$?



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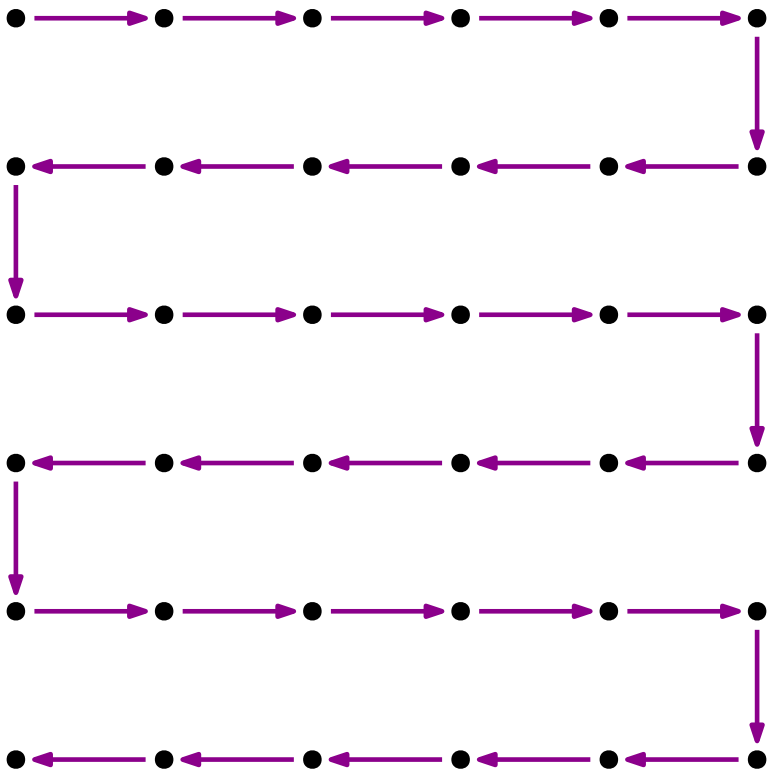

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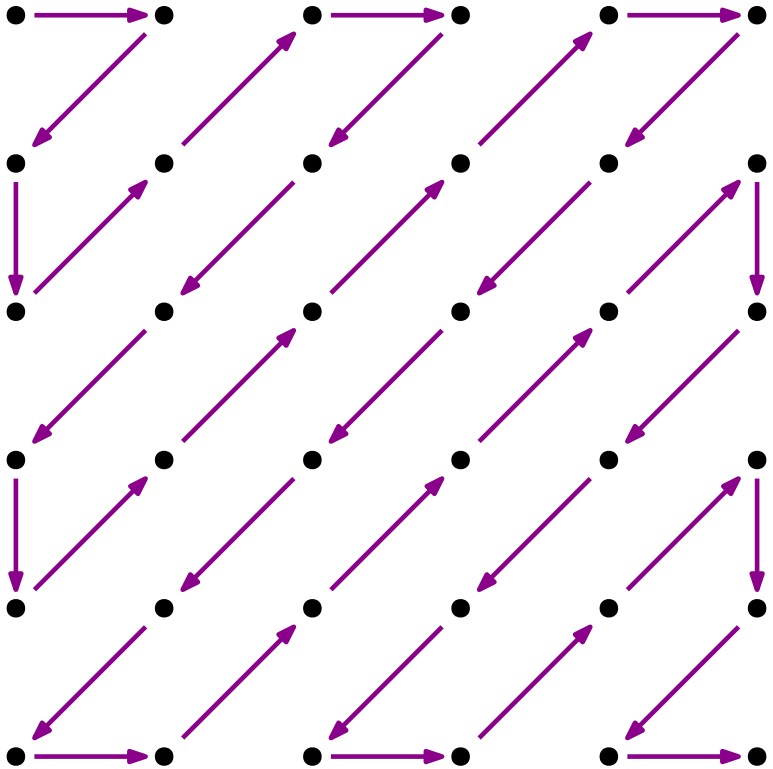

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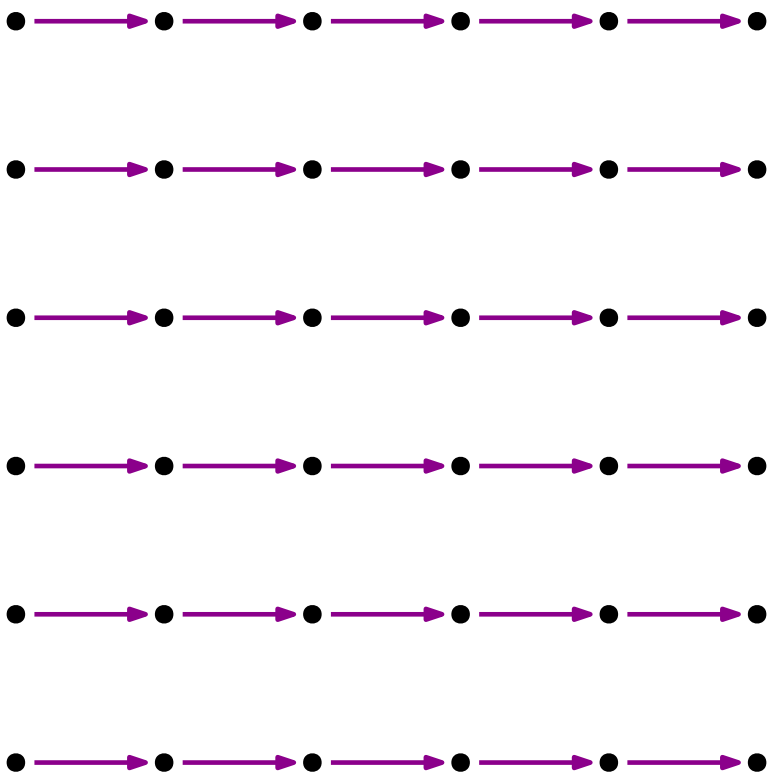

Quantization error

Theorem

Pick any $1 \leq \ell \leq d$ *and for any* $\{a_k\}_{0\leq k\leq n}$ *with* $\|a\|_\infty \leq \mu < 1$, *let* $\{\sigma_k\}_{0\leq k\leq d}$ *be a solution to a* r*-th order* $\Sigma\Delta$ *scheme applied only in the* ℓ*-th direction,*

$$a - \sigma = (\Delta_\ell)^r u.$$

Then we have

$$\left| \sum_{0\leq k\leq n} a_k p_{n,k}(x) - \sum_{0\leq k\leq n} \sigma_k p_{n,k}(x) \right| \lesssim_{r,\mu} \min\left(1, n^{-r/2} x_\ell^{-r} (1-x_\ell)^{-r}\right).$$

Relies on stability of one-dimension r-th order $\Sigma\Delta$ schemes

[Daubechies, Devore 03], [Güntürk 03].

Iterated Bernstein

Saturation: If $f \in C^2([0,1])$, then

$$\lim_{n\to\infty} n(B_n(f)(x) - f(x)) = \frac{f''(x)x(1-x)}{2}.$$

How to get around the saturation problem?

Iterated Bernstein

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How to get around the saturation problem?

Classical work: [Micchelli 73], [Felbecker 79] Define the iterated Bernstein operator

$$U_{n,m}(f) := \left(I - (I - B_n)^m\right)(f).$$

The for every $f \in C^s([0,1])$,

$$\|f - U_{n,\lceil s/2\rceil}(f)\|_\infty \lesssim_s \|f\|_{C^s} n^{-s/2}.$$

Takes advantage of smoothness, but is $U_{m,n}(f)$ a linear combination of Bernstein polynomials with small enough coefficients (necessary for quantization)?

Approximation error by linear combinations of Bernstein

Theorem

Fix any integers $d, s \geq 1$, $\delta \in (0,1)$, *and* $f \in C^s([0,1]^d)$. *Then for all integers*

$$n \geq \frac{sd^2 \|f\|_{C^2}}{4\delta},$$

there exist $\{a_k\}_{0 \leq k \leq n}$ *such that*

$$\left\| f - \sum_{0 \leq k \leq n} a_k p_{n,k} \right\|_\infty \lesssim_{s,d} \|f\|_{C^s} n^{-s/2},$$

and

$$\|a\|_\infty \leq \|f\|_\infty + \delta.$$

How to implement Bernstein polynomials?

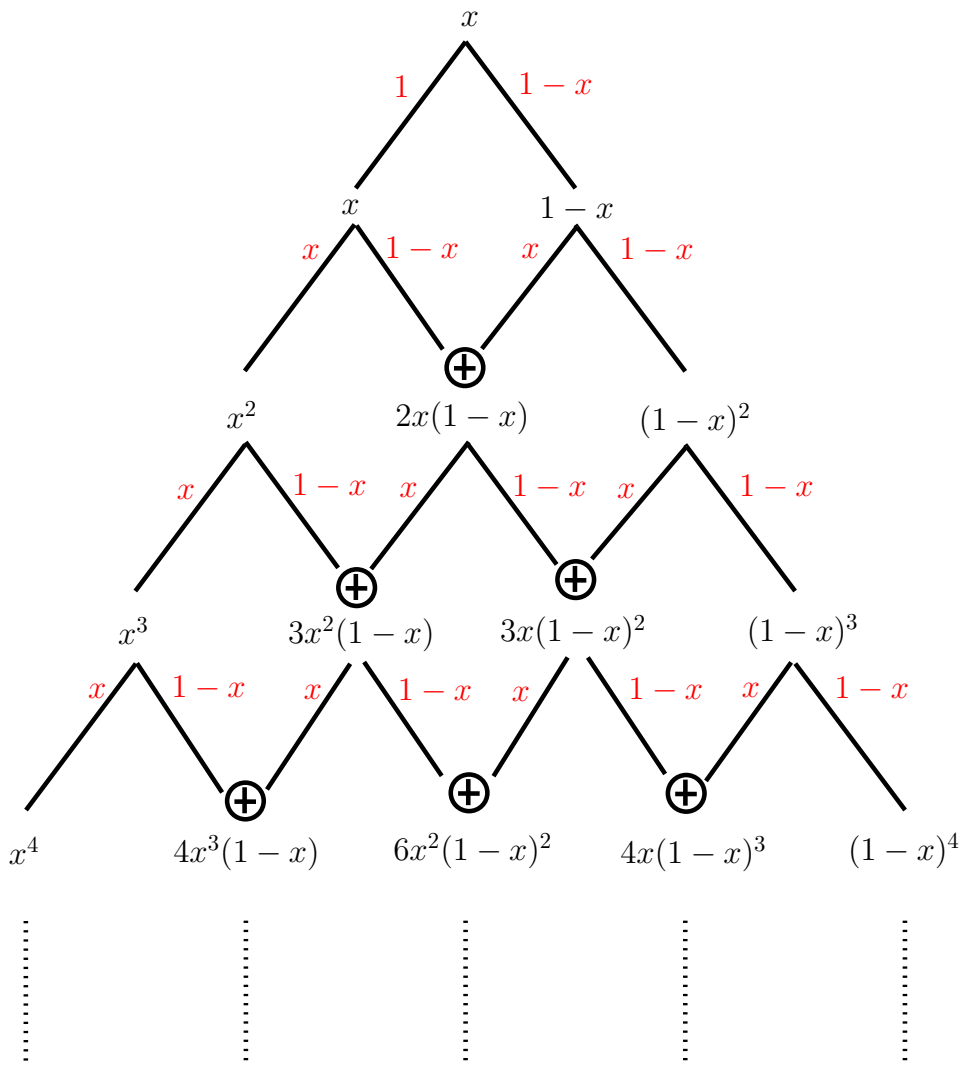

Figure: Pascal triangle implementation of Bernstein

How to multiply with coarse quantization

Key formula:

$$ab = \frac{1}{2}(a+b)^2 - \frac{1}{2}a^2 - \frac{1}{2}b^2$$

Need to find a coarsely quantized neural network that can implement squaring. Suffices for only non-negative a, b.

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- Easy if we use the quadratic non-linearity, $\rho(t) = t^2/2$, so

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- Squaring can be efficiently approximated by a ReLU network [Yarotsky 17], by exploiting the formula

  $$x(1-x) = \sum_{m=1}^{\infty} \frac{\phi^{\circ m}}{4^m}, \quad \phi(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

  Can do approximate multiplication with a $\{\pm 1/2, \pm 1, \pm 2\}$-quantized ReLU network.

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Putting everything together

$$f - f_{NN} = \underbrace{f - f_B}_{\text{Bern. approx. error}} + \underbrace{f_B - f_Q}_{\text{Bern. quan. error}} + \underbrace{f_Q - f_{NN}}_{\text{Bern. implementation error}}.$$

- $f_B =$ iterated Bernstein of f, then covered to linear Bernstein.
- $f_Q =$ function obtained from r-th order $\Sigma\Delta$ applied to coefficients of f_B.
- $f_{NN} =$ coarsely-quantized neural network approximation of f_Q.

Theorem

Fix any integers $s, d \geq 1$, $\mu \in (0,1)$, *and any* $f \in C^s([0,1]^d)$ *with* $\|f\|_\infty \leq \mu$.

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$$n \geq \frac{sd^2 \|f\|_{C^2}}{2(1-\mu)},$$

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$$|f(x) - f_{NN}(x)| \lesssim_{s,d,\mu} \|f\|_{C^s} \min\left(1, n^{-s/2} x_\ell^{-s} (1-x_\ell)^{-s}\right).$$

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$$n \geq \frac{sd^2 \|f\|_{C^2}}{2(1-\mu)},$$

there exists a function f_{NN} that is implementable by a coarsely quantized neural network such that for all $x \in [0,1]^d$,

$$|f(x) - f_{NN}(x)| \lesssim_{s,d,\mu} \|f\|_{C^s} \min\left(1, n^{-s/2} x_\ell^{-s} (1-x_\ell)^{-s}\right).$$

f_{NN} can be chosen as either

- *a $\{\pm 1\}$-quantized quadratic neural network with*

$$O(n+d) \quad \text{layers},$$
$$O(dn^2 + dn^d) \quad \text{nodes and parameters}.$$

- *a $\{\pm 1/2, \pm 1, \pm 2\}$-quantized ReLU neural network with*

$$O\Big((n+d)\big(d+s/2\big)\log n\Big) \quad \text{layers},$$
$$O\Big((n^2+dn^d)\big(d+s/2\big)\log n\Big) \quad \text{nodes and parameters}.$$

Network in the main theorem

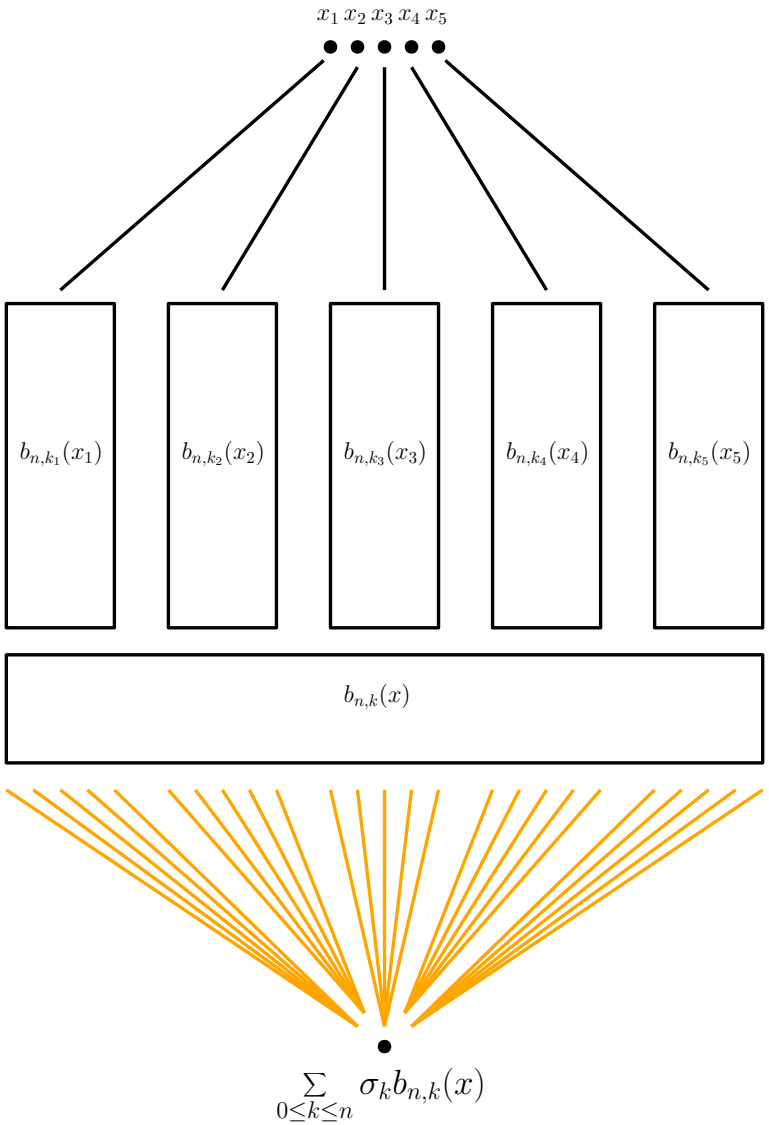

Figure: Coarsely-quantized network

Binary Bernstein Algorithm

System parameters and assumptions:

- d, dimension or number of variables,
- s, smoothness of the target function,
- $\mu \in (0,1)$, upper bound on $\|f\|_\infty$,
- n, any integer at least $sd^2\|f\|_{C^2}/(2(1-\mu))$.

Algorithm:

- Input samples $\{f(k/n)\}_{0\le k\le n}$.
- Calculate $\{a_k\}_{0\le k\le n}$ defined to be

$$a_k := U_{n,\lceil s/2\rceil}(f)\left(\frac{k}{n}\right).$$

- Run s-th order $\Sigma\Delta$ quantization in the ℓ-th direction on $\{a_k\}_{0\le k\le n}$ to obtain

$$\{\sigma_k\}_{0\le k\le n} \subset \{\pm 1\}.$$

- Guaranteed that

$$\left|f(x) - \sum_{0\le k\le n} \sigma_k p_{n,k}(x)\right| \lesssim_{s,d,\mu} \|f\|_{C^s} \min\left(1, n^{-s/2} x_\ell^{-s}(1-x_\ell)^{-s}\right).$$

Open question

Can these results be adapted and generalized to other approximation systems?

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Some properties that we used:

- Bernstein polynomials are non-negative and form a partition of unity.
- Growth conditions on central moments.
- Bernstein polynomials are localized but not too much.
- Explicit formulas for $p_{n,k+1} - p_{n,k}$; in particular, there is significant cancellation.

Binary encoding of a neural network

Given a trained neural network F we can query it at lattice points $\{F(k/n)\}_{0\le k\le n}$ and use our algorithm to produce a one-bit coefficients

$$\{\sigma_k\}_{0\le k\le n} \subseteq \{\pm 1\}.$$

(Can even be computed in parallel.)

References

Figure: Sinan Güntürk

References:

- Approximation with one-bit polynomials in Bernstein form.
  arXiv:2112.09183
- Approximation of functions with one-bit neural networks.
  arXiv:2112.09181

Can also be found on my webpage:

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Thank you!!